Uniform Lyndon Interpolation for $\mathbf{N}^+\mathbf{A}_{m,n}$

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A PDF is available!

The slides are available online at:

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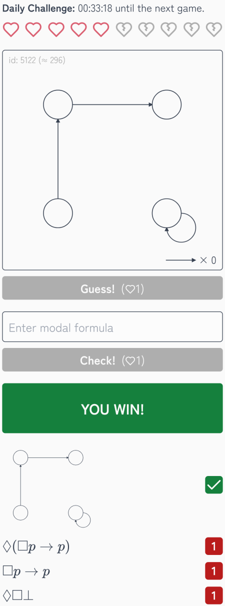
(will be displayed again at the end)

Bonus: the Kripke game!



I made a Wordle-like game where you guess the shape of a Kripke frame, just with formulas. Give it a try!

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Outline

I proved that the logic $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoys

Uniform Lyndon interpolation property,

with a new method called propositionalization.

This talk is based on:

Yuta Sato. Uniform Lyndon interpolation for the pure logic of necessitation with a modal reduction principle. *Journal of Logic and Computation*, to appear. arXiv:2503.10176.

Table of contents

The Logic $\mathrm{N}^{+}\mathbf{A}_{m,n}$

What is $\mathrm{N}^{+}\mathrm{A}_{m,n}$?

$$\mathbf{N} \;:=\; \mathbf{Cl} + \frac{\varphi}{\Box\varphi}$$

$$\mathbf{N}^{+}\mathbf{A}_{m,n} \;:=\; \mathbf{N} \;+\; \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} \;+\; \Box^{n}\varphi \to \Box^{m}\varphi$$

- $\mathbf{Cl}$: the classical propositional logic
- $\mathbf{N}$: the pure logic of necessitation (Fitting et al. 1992)
  - also obtained from the logic $\mathbf{K}$ by removing its K axiom
- $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$: required by the semantics*
- $\Box^{n}\varphi \to \Box^{m}\varphi$: a generalized reflexivity/transitivity axiom

*the completeness does not hold without it. no deep dive today

$\mathbf{N}^+\mathbf{A}_{m,n}$ vs. normal modal logics

Fact (Kurahashi and S.)

$$\mathbf{N}^+\mathbf{A}_{m,n} \subsetneq \mathbf{K} + \Box^n\varphi \to \Box^m\varphi$$

Proof.

The rule $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ is admissible in $\mathbf{K}$. The rest is trivial. $\Box$

Fact (Kurahashi and S.)

$\mathbf{N}^+\mathbf{A}_{m,n}$ has the finite frame property (ffp) for every $m, n \in \mathbb{N}$

It is still unknown to this day whether $\mathbf{K} + \Box^n\varphi \to \Box^m\varphi$ has ffp

➡ The lack of the K axiom is indeed a massive difference

The sequent calculus for $\mathrm{N^+A}_{m,n}$

A sequent calculus $\mathbf{G}_{\mathbf{N^+A}_{m,n}}$ is obtained from $\mathbf{LK}$ by adding:

$$\frac{\Rightarrow\ \varphi}{\Rightarrow\ \Box\varphi}\,(\mathrm{nec})$$

$$\frac{\Box\varphi\ \Rightarrow}{\Box\Box\varphi\ \Rightarrow}\,(\text{rosbox, when } m = 0 \text{ and } n \geq 2)$$

$$\frac{\Box^m\varphi, \Box^n\varphi, \Gamma\ \Rightarrow\ \Delta}{\Box^n\varphi, \Gamma\ \Rightarrow\ \Delta}\,(\text{accL, when } n > m)$$

$$\frac{\Gamma\ \Rightarrow\ \Delta, \Box^m\varphi, \Box^n\varphi}{\Gamma\ \Rightarrow\ \Delta, \Box^m\varphi}\,(\text{accR, when } m > n)$$

Proposition (S.)

- $\mathbf{G}_{\mathbf{N^+A}_{m,n}}$ proves $\Gamma \Rightarrow \Delta$ iff $\mathbf{N^+A}_{m,n}$ proves $\bigwedge\Gamma \to \bigvee\Delta$
- $\mathbf{G}_{\mathbf{N^+A}_{m,n}}$ admits cut elimination

Uniform Lyndon Interpolation Property

CIP and LIP (1/2)

Let $\mathrm{V}^+(\varphi)$ and $\mathrm{V}^-(\varphi)$ denote the set of variables that occur in φ positively and negatively, resp. Let also $\mathrm{V}(\varphi) = \mathrm{V}^+(\varphi) \cup \mathrm{V}^-(\varphi)$.

Example

$\mathrm{V}^+(\psi \to \chi) = \mathrm{V}^-(\psi) \cup \mathrm{V}^+(\chi),\ \mathrm{V}^-(\psi \to \chi) = \mathrm{V}^+(\psi) \cup \mathrm{V}^-(\chi)$

L is said to enjoy Craig interpolation property (CIP) if for every φ, ψ s.t. $L \vdash \varphi \to \psi$, there is χ s.t.:

1. $\mathrm{V}(\chi) \subseteq \mathrm{V}(\varphi) \cap \mathrm{V}(\psi)$;
2. $L \vdash \varphi \to \chi$ and $L \vdash \chi \to \psi$.

Such χ is called an interpolant of $\varphi \to \psi$ in L.

CIP and LIP (2/2)

Let $\mathrm{V}^{+}(\varphi)$ and $\mathrm{V}^{-}(\varphi)$ denote the set of variables that occur in φ positively and negatively, resp. Let also $\mathrm{V}(\varphi) = \mathrm{V}^{+}(\varphi) \cup \mathrm{V}^{-}(\varphi)$.

Example

$$\mathrm{V}^{+}(\psi \to \chi) = \mathrm{V}^{-}(\psi) \cup \mathrm{V}^{+}(\chi),\ \mathrm{V}^{-}(\psi \to \chi) = \mathrm{V}^{+}(\psi) \cup \mathrm{V}^{-}(\chi)$$

L is said to enjoy Lyndon interpolation property (LIP) if for every φ, ψ s.t. $L \vdash \varphi \to \psi$, there is χ s.t.:

1. $\mathrm{V}^{\bullet}(\chi) \subseteq \mathrm{V}^{\bullet}(\varphi) \cap \mathrm{V}^{\bullet}(\psi)$ $(\bullet \in \{+, -\})$;
2. $L \vdash \varphi \to \chi$ and $L \vdash \chi \to \psi$.

Such χ is called an interpolant of $\varphi \to \psi$ in L.

UIP and ULIP (1/2)

L is said to enjoy Uniform interpolation property (UIP) if for any φ and any finite set of variables P, there is χ s.t.

1. $\mathrm{V}(\chi) \subseteq \mathrm{V}(\varphi) \setminus P$;
2. $L \vdash \varphi \to \chi$;
3. $L \vdash \chi \to \psi$ for any ψ s.t. $L \vdash \varphi \to \psi$ and $\mathrm{V}(\psi) \cap P = \emptyset$.

Such χ is called a post-interpolant of (φ, P) in L.

UIP and ULIP (2/2)

L is said to enjoy Uniform Lyndon interpolation property (ULIP) if for any φ and any finite sets of variables P^+, P^-, there is χ s.t.

1. $\mathrm{V}^\bullet(\chi) \subseteq \mathrm{V}^\bullet(\varphi) \setminus P^\bullet$ $(\bullet \in \{+, -\})$;
2. $L \vdash \varphi \to \chi$;
3. $L \vdash \chi \to \psi$ for any ψ s.t. $L \vdash \varphi \to \psi$ and $\mathrm{V}^\bullet(\psi) \cap P^\bullet = \emptyset$ $(\bullet \in \{+, -\})$.

Such χ is called a post-interpolant of (φ, P^+, P^-) in L.

Several facts on the interpolation properties (1/2)

Fact

- If L has UIP, then L has CIP
- If L has LIP, then L has CIP
- If L has ULIP, then L has both UIP and LIP (Kurahashi 2020)

Fact (Kurahashi 2020)

- The classical propositional logic $\mathbf{Cl}$ enjoys ULIP
- The modal logic $\mathbf{K}$ enjoys ULIP

Several facts on the interpolation properties (2/2)

The situation is complicated for the extensions of $\mathbf{K}$:

Fact

- $\mathbf{KT} = \mathbf{K} + \Box\varphi \to \varphi$ enjoys ULIP (Kurahashi 2020)
- For $m > 0$, $\mathbf{K} + \Box\varphi \to \Box^m\varphi$ enjoys CIP (Gabbay 1972) and LIP (Kuznets 2016)
- $\mathbf{K4} = \mathbf{K} + \Box\varphi \to \Box\Box\varphi$ lacks UIP (Bílková 2007)
- $\mathbf{K} + \Box\Box\varphi \to \Box\varphi$ lacks even CIP (Marx 1995)

$\mathbf{K} + \Box^n\varphi \to \Box^m\varphi$, in general, may or may not enjoy them

➡ What happens if we weaken it to $\mathbf{N}^+\mathbf{A}_{m,n}$?

The Propositionalization Method

Propositionalization, in short

ULIP of a logic is sometimes proven by embedding it to some weaker logic where ULIP is already known:

Example

Through the boxdot translation, ULIP of $\mathbf{K}$ implies ULIP of $\mathbf{KT}$, and the failure of it in $\mathbf{S4}$ implies that of $\mathbf{K4}$

I gave a sufficient condition on such embeddings:

Theorem (S.)

For any logics $L \subseteq M$, if there is a translation with certain properties, propositionalization, of M into L, and L has ULIP, then so does M

Propositionalization, in detail (1/3)

Given a logic X, let $\mathscr{L}_X$ designate the language of X.

Consider logics L and M s.t. $\mathscr{L}_L \subseteq \mathscr{L}_M$ and $L \subseteq M$.

Now we want to *propositionalize* any $\mathscr{L}_M$-formula that is not expressible in $\mathscr{L}_L$:

Definition

Let L' be the same logic as L, but its propositional variables extended by adding a fresh one p_φ for every $\varphi \in \mathscr{L}_M$.

Definition

Let $\sigma : \mathscr{L}_{L'} \to \mathscr{L}_M$ be the substitution that replaces every p_φ with φ, then $L' \vdash \rho$ implies $M \vdash \sigma(\rho)$ for any $\rho \in \mathscr{L}_{L'}$.

Propositionalization, in detail (2/3)

Definition

A pair of translations $\sharp, \flat : \mathscr{L}_M \to \mathscr{L}_{L'}$ is called a propositionalization of M into L if the following are met:

(Embeddable) $M \vdash \varphi \to \psi$ implies $L' \vdash \varphi^\flat \to \psi^\sharp$;

(Invertible) $M \vdash \sigma(\varphi^\sharp) \to \varphi$ and $M \vdash \varphi \to \sigma(\varphi^\flat)$;

(Polarity-preserving) For $(\bullet, \circ) \in \{(+,-),(-,+)\}$, $\natural \in \{\sharp, \flat\}$:

- $p \in \mathrm{V}^\bullet(\varphi^\natural)$ implies $p \in \mathrm{V}^\bullet(\varphi)$;
- $p_\psi \in \mathrm{V}^\bullet(\varphi^\natural)$ implies $\mathrm{V}^\bullet(\psi) \subseteq \mathrm{V}^\bullet(\varphi)$, $\mathrm{V}^\circ(\psi) \subseteq \mathrm{V}^\circ(\varphi)$.

Propositionalization, in detail (3/3)

Theorem (S.)

If there is a propositionalization $(\sharp, \flat)$ of M into L, and L has ULIP, then M does also

Proof (outline).

Take any φ, P^+, P^-. We extend $P^\bullet$ to $Q^\bullet$ by adding every *problematic*[†] p_ψ found in $\varphi^\flat$. By ULIP of L, we get a post-interpolant χ' of $(\varphi^\flat, Q^+, Q^-)$. Then, embeddability, invertibility, and polarity-preservingness of $\sharp, \flat$ assert that $\chi = \sigma(\chi')$ is indeed a post-interpolant of (φ, P^+, P^-) in M. □

[†] the actual condition for p_ψ to be problematic is very complicated

The Main Theorem

The Main Theorem

Theorem (S.)

There is a propositionalization $(\sharp, \flat)$ of $\mathbf{N}^{+}\mathbf{A}_{m,n}$ into $\mathbf{Cl}$

Proof (outline).

We construct such $\sharp, \flat$ that a cut-free proof of $\Gamma \Rightarrow \Delta$ in $\mathbf{G}_{\mathbf{N}^{+}\mathbf{A}_{m,n}}$ can be *emulated* as a proof of $\Gamma^{\flat} \Rightarrow \Delta^{\sharp}$ in $\mathbf{LK}$, then embeddability natually holds. We also ensure invertibility and polarity-preservingness by adding just the right amount of information to enable such emulation. $\square$

Corollary

$\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoys ULIP!

Summing it up (1/2)

It is known that $\mathbf{K} + \Box^n\varphi \to \Box^m\varphi$ does <u>not</u>, in general, enjoy all of CIP, LIP, UIP, and ULIP:

- $\mathbf{K4} = \mathbf{K} + \Box\varphi \to \Box\Box\varphi$ lacks UIP
- $\mathbf{K} + \Box\Box\varphi \to \Box\varphi$ lacks even CIP

However, $\mathbf{N}^+\mathbf{A}_{m,n}$ enjoy all of them for every $m, n \in \mathbb{N}$!

Summing it up (1/2)

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However, $\mathbf{N}^+\mathbf{A}_{m,n}$ enjoy all of them for every $m, n \in \mathbb{N}$!

Open Problem

To what extent the presence of the K axiom is *harmful* for a logic in terms of interpolation properties?

- Is there a logic between $\mathbf{N4}$ and $\mathbf{K4}$ that lacks UIP?
- Is there a logic between $\mathbf{N} + \Box\Box\varphi \to \Box\varphi$ and $\mathbf{K} + \Box\Box\varphi \to \Box\varphi$ that lacks CIP?

Summing it up (2/2)

We also developed a general method for proving ULIP:

Theorem (S.)

For any logics $L \subseteq M$, if there is a propositionalization of M into L, and L has ULIP, then so does M

Summing it up (2/2)

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Theorem (S.)

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Open Problems

- Can we possibly say that if ULIP holds, then some nontrivial propositionalization exists? For example, can we construct propositionalizations of $\mathbf{K}$ into $\mathbf{N}$ or $\mathbf{Cl}$?
- Can we characterize a syntactic property on sequent calculi that corresponds to the existence of a propositionalization? (e.g. Iemhoff 2019, Akbar Tabatabai & Jalali 2025)

Thanks!

That's all!

The Slides

cannorin.net/math/alc2025.pdf

The Kripke Game

cannorin.net/kripke

Appendix & References

Why the accL and accR rules look like that?

You may be wondering why we did not just use an initial sequent $\Box^n \varphi \Rightarrow \Box^m \varphi$ to represent the axiom $\Box^n \varphi \to \Box^m \varphi$.

Suppose $m > 0$ and $n = 0$, and consider the sequent calculus obtained from $\mathbf{LK}$ by adding the nec rule and the said initial sequent. This would permit the following cut, which cannot be eliminated:

$$\dfrac{\dfrac{\varphi_1 \;\Rightarrow\; \varphi_1}{\varphi_1 \;\Rightarrow\; \varphi_1 \vee \varphi_2}\,(\vee\mathrm{R}) \qquad \varphi_1 \vee \varphi_2 \;\Rightarrow\; \Box^m(\varphi_1 \vee \varphi_2)}{\varphi_1 \;\Rightarrow\; \Box^m(\varphi_1 \vee \varphi_2)}\,\text{(cut)}$$

The same problem happens for the case when $m = 0$ and $n > 0$.

How do we get $Q^\bullet$ from $P^\bullet$?

Basically, we want to add p_ψ to $Q^\bullet$ if $\mathrm{V}(\psi)$ overlap with $P^\bullet$. We need to be extra careful here; if $p \in \mathrm{V}^-(\psi)$ and $p_\psi \in \mathrm{V}^-(\varphi^\flat)$, then it must be that $p \in \mathrm{V}^+(\varphi^\flat)$.

Definition

Let us say $\psi \in \mathscr{L}_M$ is $+$-safe if $P^+ \cap \mathrm{V}^+(\psi) = P^- \cap \mathrm{V}^-(\psi) = \emptyset$, and is $-$-safe if $P^+ \cap \mathrm{V}^-(\psi) = P^- \cap \mathrm{V}^+(\psi) = \emptyset$.

For $\bullet \in \{+, -\}$, we let:

$$Q^\bullet = P^\bullet \cup \left\{ p_\psi \in \mathrm{V}(\varphi^\flat) \;\middle|\; \psi \text{ is not } \bullet\text{-safe} \right\}.$$

Why cut elimination is needed for propositionalization?

First, embeddability of $\sharp, \flat$ (if $M \vdash \varphi \to \psi$, then $L' \vdash \varphi^\flat \to \psi^\sharp$) implies that $L' \vdash \varphi^\flat \to \varphi^\sharp$. So $\varphi^\sharp$ is, in general, provably weaker than $\varphi^\flat$.

Now suppose that $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ were obtained by the cut rule:

$$\frac{\Gamma_1 \ \Rightarrow \ \Delta_1, \varphi \qquad \varphi, \Gamma_2 \ \Rightarrow \ \Delta_2}{\Gamma_1, \Gamma_2 \ \Rightarrow \ \Delta_1, \Delta_2} \text{(cut)}$$

Then by the induction hypothesis, $\Gamma_1^\flat \Rightarrow \Delta_1^\sharp, \varphi^\sharp$ and $\varphi^\flat, \Gamma_2^\flat \Rightarrow \Delta_2^\sharp$ would be provable in the sequent calculus for L.

As $\varphi^\sharp$ is weaker than $\varphi^\flat$, there would be no way of applying the cut rule to these two sequents and thus obtaining $\Gamma_1^\flat, \Gamma_2^\flat \Rightarrow \Delta_1^\sharp, \Delta_2^\sharp$.

References

This talk is based on the paper indicated by $\star$:

- $\star$ Yuta Sato. Uniform Lyndon interpolation for the pure logic of necessitation with a modal reduction principle. *Journal of Logic and Computation*, to appear. arXiv:2503.10176.
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